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On Dual Functionals of Polynomials in B-Form

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We consider the space of multivariate polynomials in B-form and give an explicit representation of its dual space. Using this space, the B-form of a monomial can be easily obtained. We also derive a simple algorithm to convert a polynomial from its Taylor expansion to its B-form. Each basis element of the dual space is bounded, and we give explicit upper bounds. These upper bounds can be used to improve some known estimates by giving explicit constants.

Polynomials in B-form, i.e., polynomials in Bernstein, Bézier, or de Casteljau representation, were originally used in car body design. Later, this form for polynomials was widely studied in computer aided geometry design and became an important tool for representation and computation of polynomial and spline curves and surfaces, see [2, 10, 11]. Recently, this important tool has been adopted and developed as a powerful tool in multivariate spline approximation, see, e.g., [4, 6]. The theoretical and practical aspects of the study of polynomials in B-form can be found in [3, 9, 12]. Many basic properties of the B-form have been studied. Readers are referred to the references mentioned above.

One of the important subjects on the B-form which is neglected in the above references is the dual space of polynomials in B-form. In the univariate setting this topic was considered in [1]. In the multivariate case, the subject has been studied only briefly in [13]. In fact, the formulation of a dual basis in [13] for the special bivariate case can essentially be found in [7]. It is our primary concern in this paper to extend the study of dual functionals to the multivariate setting. We will study the linear functionals L_α defined on $\mathbb{P}_n$, the space of polynomials of total degree $\leqslant n$ in s variables, satisfying

$$L_\alpha B_\gamma = \delta_{\alpha\gamma} = \begin{cases} 1, & \alpha = \gamma; \\ 0, & \alpha \neq \gamma, \end{cases} \qquad \forall\ |\alpha| = |\gamma| = n,$$

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where $\{B_\gamma, |\gamma|=n\}$ denotes a basis of $\mathbb{P}_n$ (the precise definition will be given below).

In the following, we will give an explicit formulation of dual functionals satisfying this condition which will include the dual functional basis in [13] as a special case. With the dual functionals, we are able to express monomials in B-form and convert polynomials from its Taylor expansion to its B-form. We will explore the connection of these dual functionals to some of the basic properties of the B-form. These dual functionals are bounded on $\mathbb{P}_n$. This fact will be used to improve some known estimates by specifying explicit constants.

Let us first briefly introduce the B-form for polynomials.

Denote by $\mathbf{v}^0, ..., \mathbf{v}^s$ the vertices of an s-simplex $T=\langle \mathbf{v}^0, ..., \mathbf{v}^s\rangle = \{\mathbf{x}=\sum_{i=0}^s \lambda_i \mathbf{v}^i : \sum_{i=0}^s \lambda_i = 1,\ \lambda_i \geqslant 0\} \subseteq \mathbb{R}^s$. Here, an s-simplex is the convex hull of its vertices with positive volume. Setting

$$B_\alpha(\lambda)=\frac{|\alpha|!}{\alpha!}\lambda^\alpha, \qquad \forall \alpha \in \mathbb{Z}_+^{s+1},$$

we know that $\{B_\alpha(\lambda), |\alpha|=n\}$ forms a basis of $\mathbb{P}_n$, where $|\alpha|=\sum_{i=0}^s \alpha_i$ and $\alpha! = \alpha_0! \cdots \alpha_s!$ as usual. Hence, any $p_n \in \mathbb{P}_n$ can be expressed in the form

$$p_n(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha B_\alpha(\lambda)$$

which is called the polynomial in B-form (with respect to T).

Now we need some notation and definitions in order to introduce the dual functionals.

Define D_{ij}, the derivative in the direction $\langle \mathbf{v}^i, \mathbf{v}^j\rangle$, by

$$D_{ij} f(\mathbf{x}) = \lim_{t\to 0} \frac{f(\mathbf{x}+t(\mathbf{v}^i-\mathbf{v}^j))-f(\mathbf{x})}{t}.$$

Consequently, $D_{ij}^k = D_{ij}^{k-1} D_{ij}$ for $k \geqslant 2$. Also,

$$D_i^\beta = D_{i0}^{\beta_1} \cdots D_{i,i-1}^{\beta_i} D_{i,i+1}^{\beta_{i+1}} \cdots D_{is}^{\beta_s}, \qquad \forall \beta \in \mathbb{Z}_+^s.$$

Similarly, denote by Δ_{ij} the difference operator

$$\Delta_{ij} c_\alpha = c_{\alpha+e^i} - c_{\alpha+e^j}, \qquad \forall \alpha \in \mathbb{Z}_+^{s+1},$$

where e^i, e^j are the standard unit vectors in $\mathbb{R}^{s+1}$ and $0 \leqslant i, j \leqslant s$. We also denote

$$\Delta_i^\beta = \Delta_{i0}^{\beta_1} \cdots \Delta_{i,i-1}^{\beta_i} \Delta_{i,i+1}^{\beta_{i+1}} \cdots \Delta_{is}^{\beta_s}.$$

It is known that

$$D_{ij} p_n(\mathbf{x}) = n \sum_{|\alpha|=n-1} \Delta_{ij} c_\alpha B_\alpha(\lambda).$$

(Cf., e.g., [9].) Thus,

$$D_i^\beta p_n(\mathbf{x}) = \frac{n!}{(n-|\beta|)!} \sum_{|\alpha|=n-|\beta|} \Delta_i^\beta c_\alpha B_\alpha(\lambda).$$

An application of this differentiation formula gives the following lemma.

LEMMA 1. *For any $p_n \in \mathbb{P}_n$, and any integer $l > 0$,*

$$\sum_{|\alpha|=n+l} p_n(\mathbf{x}_\alpha)\, B_\alpha(\lambda) \in \mathbb{P}_n,$$

where $\mathbf{x}_\alpha = (1/|\alpha|) \sum_{i=0}^s \alpha_i \mathbf{v}^i$, $\forall\, |\alpha| = n+l$.

Proof. For any β such that $|\beta| > n$,

$$D_0^\beta \left(\sum_{|\alpha|=n+l} p_n(\mathbf{x}_\alpha)\, B_\alpha(\lambda) \right) = \frac{(n+l)!}{(n+l-|\beta|)!} \sum_{|\alpha|=n+l-|\beta|} \Delta_0^\beta p_n(\mathbf{x}_\alpha)\, B_\alpha(\lambda).$$

But $\Delta_0^\beta p_n(\mathbf{x}_\alpha) = 0$ $\forall \alpha$, since p_n is a polynomial of total degree $\leqslant n$. Hence,

$$D_0^\beta \left(\sum_{|\alpha|=n+l} p_n(\mathbf{x}_\alpha)\, B_\alpha(\lambda) \right) = 0 \qquad \forall\, |\beta| > n.$$

This completes the proof.

Let $\Lambda_n^{s+1} = \{\alpha \in \mathbb{Z}_+^{s+1} : |\alpha| = n\}$. A subset $M \subset \mathbb{Z}_+^s$ is called a lower set if it has the property that $\beta \in M$ implies $\gamma \in M$ for any $\gamma \leqslant \beta$.

We say that $M_0, ..., M_s \subset \mathbb{Z}_+^s$ induce a partition of Λ_n^{s+1} if they satisfy:

(i) $A_i^n M_i \cap A_j^n M_j = \varnothing$, $i \neq j$, and

(ii) $\bigcup_{i=0}^s A_i^n M_i = \Lambda_n^{s+1}$.

Here, A_i^n is the extension operator defined by

$$A_i^n \beta = (\beta_1, ..., \beta_{i-1}, n-|\beta|, \beta_{i+1}, ..., \beta_s), \qquad \forall \beta \in \mathbb{Z}_+^s,$$

$i = 0, 1, ..., s$.

Recall the following inversion formula. Let $M \subseteq \mathbb{Z}_+^s$ be a lower set, and

$$f(\alpha) = \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} (-1)^{|\gamma|} g(\gamma) \qquad \forall \alpha \in M.$$

Then

$$g(\gamma) = \sum_{\alpha \leqslant \gamma} \binom{\gamma}{\alpha} (-1)^{|\alpha|} f(\alpha) \qquad \forall \gamma \in M.$$

Here and throughout, we assume that $M_0, ..., M_s \subset \mathbb{Z}_+^s$ are lower sets and induce a partition of Λ_n^{s+1}. Define the linear functional $L_{\Lambda_i^n \beta}$ by

$$L_{\Lambda_i^n \beta} f = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} D_i^\gamma f(\mathbf{v}^i)$$

for $\beta \in M_i$, $i = 0, ..., s$. Then we claim that

$$\{L_\alpha : \alpha \in \Lambda_n^{s+1}\} := \{L_{\Lambda_i^n \beta} : \beta \in M_i, i = 0, ..., s\}$$

is a dual functional basis of $\mathbb{P}_n$ in B-form. That is,

THEOREM 1. *Suppose that $M_0, ..., M_s \subseteq \mathbb{Z}_+^s$ are all lower sets and induce a partition of Λ_n^{s+1}. Then for any $\alpha, \gamma \in \Lambda_n^{s+1}$,*

$$L_\alpha B_\gamma = \delta_{\alpha\gamma},$$

where

$$\delta_{\alpha\gamma} = \begin{cases} 1, & \alpha = \gamma \\ 0, & \alpha \neq \gamma \end{cases}$$

is the Kronecker's delta.

The primitive version of these dual functionals can be found in [8] where it was applied to solve interpolation problems. With those dual functionals, we readily have the following

COROLLARY. *Let $p_n = \sum_{|\alpha| = n} c_\alpha B_\alpha$. Then*

$$c_\alpha = L_\alpha p_n, \qquad \forall \alpha \in \Lambda_n^{s+1}.$$

Proof of Theorem 1. Fix $0 \leq i \leq s$. Let $\alpha = \Lambda_j^n \nu$, $\nu \in M_j$, and $c_\gamma^\alpha = \delta_{\alpha\gamma}$, $\gamma \in \Lambda_n^{s+1}$. Then

$$\begin{aligned} D_i^\beta B_\alpha(\mathbf{v}^i) &= D_i^\beta \sum_{|\gamma| = n} c_\gamma^\alpha B_\gamma(\lambda) \Big|_{\mathbf{v}^i} \\ &= \frac{n!}{(n-|\beta|)!} \sum_{|\gamma| = n - |\beta|} \Delta_i^\beta c_\gamma^\alpha B_\gamma(\lambda) \Big|_{\mathbf{v}^i} \\ &= \frac{n!}{(n-|\beta|)!} \Delta_i^\beta c_{(n-|\beta|) e^i}^\alpha \\ &= \frac{n!}{(n-|\beta|)!} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (-1)^{|\beta| - |\gamma|} c_{\Lambda_i^n \gamma}^\alpha. \end{aligned}$$

Hence,

$$L_{A_i^n\mu} B_{A_j^n\nu} = \sum_{\beta \leqslant \mu} \binom{\mu}{\beta} \frac{(n-|\beta|)!}{n!} D_i^\beta B_{A_j^n\nu}(\mathbf{v}^i)$$

$$= \sum_{\beta \leqslant \mu} \binom{\mu}{\beta} (-1)^{|\beta|} \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\gamma|} c^\alpha_{A_i^n\gamma}.$$

It follows that

$$L_{A_i^n\mu} B_{A_j^n\nu} = 0, \qquad \forall \nu \in M_j \quad \text{and} \quad \forall \mu \in M_i,$$

where $j \neq i$, since $c^\alpha_{A_i^n\gamma} = c^{A_j^n\nu}_{A_i^n\gamma} = 0$. If $j = i$, we let $f(\beta) = \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\gamma|} c^{A_i^n\nu}_{A_i^n\gamma}$, $\beta \in M_i$. Then by the inversion formula,

$$L_{A_i^n\mu} B_{A_i^n\nu} = \sum_{\beta \leqslant \mu} \binom{\mu}{\beta} (-1)^{|\beta|} f(\beta) = c^{A_i^n\nu}_{A_i^n\mu} = \delta_{\nu\mu}.$$

Hence, we have established the theorem.

EXAMPLE 1. Let $s = 2$ and $n = 5$. M_0, M_1, and M_2 can be chosen as follows: $M_0 = M_1 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0.2)\}$ and $M_2 = M_0 \cup \{(2,1), (1,2), (2.2)\}$. Clearly, they are lower sets and induce a partition of Λ_5^3.

Remark. We may use these dual functionals to derive the de Casteljau's Algorithm which is a very popular way to evaluate any polynomial $p_n(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha B_\alpha(\lambda)$ in computer aided geometric design. (See other derivations of de Casteljau's algorithm in [1–3, 9–11].) To evaluate p_n at $\mathbf{x} = \sum_{i=0}^s \lambda_i \mathbf{v}^i$ with $\sum_{i=0}^s \lambda_i = 1$, we may write the B-form for p_n with respect to $T' = \langle \mathbf{x}, \mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^s \rangle$, as $p_n(y) = \sum_{|\alpha|=n} b_\alpha B_\alpha(\mu)$. Here, we have assumed that $\lambda_0 > 0$ without loss of generality. Let $\beta_0 = (n, 0, ..., 0) \in \mathbb{Z}^s_+$. Then

$$p_n(\mathbf{x}) = b_{A_s^n\beta_0} = L_{A_s^n\beta_0} p_n$$

$$= \sum_{\gamma \leqslant \beta_0} \binom{\beta_0}{\gamma} \frac{(n-|\gamma|)!}{n!} D^{\gamma_1}_{\mathbf{x}-\mathbf{v}^s} p_n(\mathbf{v}^s)$$

$$= \sum_{i=0}^n \binom{n}{i} \frac{(n-i)!}{n!} \left(\sum_{j=0}^{s-1} \lambda_j D_{\mathbf{v}^j - \mathbf{v}^s} \right)^i \sum_{|\alpha|=n} c_\alpha B_\alpha \Bigg|_{\mathbf{v}^s}$$

$$= \sum_{i=0}^n \binom{n}{i} \frac{(n-i)!}{n!} \frac{n!}{(n-i)!} \left(\sum_{j=0}^{s-1} \lambda_j \Delta_{js} \right)^i c_{(0,...,0,n-i)}$$

$$= \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{s-1} \lambda_j \Delta_{js} \right)^i E_s^{n-i} c_{(0, ..., 0)}$$

$$= \left(\sum_{i=0}^{s-1} \lambda_i \Delta_{is} + E_s \right)^n c_{(0, ..., 0)}$$

$$= \left(\sum_{i=0}^{s} \lambda_i E_i \right)^n c_{(0, ..., 0)}$$

$$= \left(\sum_{i=0}^{s} \lambda_i E_i \right)^{n-k} \left(\sum_{i=0}^{s} \lambda_i E_i \right)^k c_{(0, ..., 0)}$$

$$= \sum_{|\beta|=k} \left(\sum_{i=0}^{s} \lambda_i E_i \right)^{n-k} c_\beta B_\beta(\lambda),$$

where E_i denotes the shift operator defined by

$$E_i c_\alpha = c_{\alpha+e^i}, \qquad \alpha \in \mathbb{Z}_+^s, \qquad i = 0, ..., s.$$

Hence, for $|\beta| = n$, set $P_\beta(\lambda) = c_\beta$. Then for $|\beta| = n-1$, $n = 2, ..., 0$, compute

$$P_\beta(\lambda) = \sum_{i=0}^{s} \lambda_i P_{\beta+e^i}(\lambda);$$

to obtain $P_0(\lambda) = (\sum_{i=0}^{s} \lambda_i E_i)^n c_{(0, ..., 0)} = p_n(\mathbf{x})$. This is the so-called de Casteljau's algorithm.

Similarly, any b_α of p_n with respect to T' can be obtained as well. For any α with $|\alpha| = n$, let $\beta \in \mathbb{Z}_+^s$ such that $\alpha = A_s^n \beta$. Then

$$b_\alpha = L_{A_s^n \beta} p_n = \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} D_s^\gamma p_n(\mathbf{v}^s)$$

$$= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} (D_{\mathbf{x}-\mathbf{v}^s})^{\gamma_1} (D_{\mathbf{v}^1-\mathbf{v}^s})^{\gamma_2} \cdots (D_{\mathbf{v}^{s-1}-\mathbf{v}^s})^{\gamma_s} p_n(\mathbf{v}^s)$$

$$= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{(n-|\gamma|)!}{n!} \left(\sum_{j=0}^{s-1} \lambda_j D_{\mathbf{v}^j-\mathbf{v}^s} \right)^{\gamma_1}$$

$$\times (D_{\mathbf{v}^1-\mathbf{v}^s})^{\gamma_2} \cdots (D_{\mathbf{v}^{s-1}-\mathbf{v}^s})^{\gamma_s} \sum_{|\eta|=n} c_\eta B_\eta \bigg|_{\mathbf{v}^s}$$

$$= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \left(\sum_{j=0}^{s-1} \lambda_j \Delta_{js} \right)^{\gamma_1} \Delta_{1s}^{\gamma_2} \cdots \Delta_{s-1,s}^{\gamma_s} c_{(0, ..., 0, n-|\gamma|)}$$

$$= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \left(\sum_{j=0}^{s-1} \lambda_j \Delta_{js} \right)^{\gamma_1} \Delta_{1s}^{\gamma_2} \cdots \Delta_{s-1,s}^{\gamma_s} E_s^{n-|\gamma|} c_{(0, ..., 0)}$$

$$= \left(\sum_{j=0}^{s-1} \lambda_j \Delta_{js} + E_s \right)^{\beta_1} (\Delta_{1s} + E_s)^{\beta_2} \cdots (\Delta_{s-1,s} + E_s)^{\beta_s} E_s^{n-|\beta|} c_{(0,\dots,0)}$$

$$= \left(\sum_{j=0}^{s} \lambda_j E_j \right)^{\beta_1} E_1^{\beta_2} \cdots E_{s-1}^{\beta_s} E_s^{n-|\beta|} c_{(0,\dots,0)}$$

$$= \left(\sum_{j=0}^{s} \lambda_j E_j \right)^{\alpha_0} c_{\alpha - \alpha_0 e^0} = P_{\alpha - \alpha_0 e^0}(\lambda)$$

because

$$P_{\alpha - \alpha_0 e^0}(\lambda) = \sum_{j=0}^{s} \lambda_j P_{\alpha - \alpha_0 e^0 + e^j}(\lambda)$$

$$= \cdots = \underbrace{\sum_{j=0}^{s} \lambda_j \cdots \sum_{i=0}^{s}}_{\alpha_0} \lambda_i P_{\alpha - \alpha_0 e^0 + \underbrace{e^j + \cdots + e^i}_{\alpha_0}}$$

$$= \underbrace{\sum_{j=0}^{s} \lambda_j \cdots \sum_{i=0}^{s}}_{\alpha_0} \lambda_i c_{\alpha - \alpha_0 e^0 + \underbrace{e^j + \cdots + e^i}_{\alpha_0}}$$

$$= \sum_{j=0}^{s} \lambda_j \cdots \sum_{i=0}^{s} \lambda_i E_j \cdots E_i c_{\alpha - \alpha_0 e^0}$$

$$= \left(\sum_{j=0}^{s} \lambda_j E_j \right)^{\alpha_0} c_{\alpha - \alpha_0 e^0}.$$

Therefore, the other b_α's of p_n with respect to T' are byproducts of using de Casteljau's algorithm to find $p_n(\mathbf{x})$ $(= b_{(n,0,\dots,0)})$.

Remark on the B-Net Subdivision Algorithm. Suppose that $\{c_\alpha\}_{|\alpha|=n}$ are the given B-coefficients of a polynomial p_n with respect to T. Here $s=2$. We split T into two subtriangles by connecting $\mathbf{v}^0$ and the midpoint of $\mathbf{v}^1$ and $\mathbf{v}^2$. Then the B-coefficients of p_n with respect to each of these two subtriangles can be found simultaneously by applying de Casteljau's algorithm at the midpoint $(\mathbf{v}^1 + \mathbf{v}^2)/2$ as derived above. Suppose now that we split each of the two subtriangles into two by connecting the midpoint $(\mathbf{v}^1 + \mathbf{v}^2)/2$ to the midpoints of its opposite edges, and so on. (Cf. [9].) This procedure is called B-net subdivision. It is worth noticing that in this particular example the multiplications of an application of de Casteljau's algorithm are just binary shifts. Hence the B-net subdivision algorithm is very efficient.

By using the dual functionals $\{L_\alpha\colon |\alpha|=n\}$, the polynomial interpolating given data $\{f_{i\beta}\colon \beta\in M_i, i=0, ..., s\}$ in the sense that

$$D_i^\beta p_n(f, \mathbf{v}^i)=f_{i\beta}, \qquad \beta\in M_i, \quad i=0, ..., s$$

can be easily found.

THEOREM 2. *Suppose that $M_0, ..., M_s\subseteq \mathbb{Z}_+^s$ are all lower sets and induce a partition of Λ_n^{s+1}. Then for any given data $\{f_{i\beta}\colon \beta\in M_i, i=0, ..., s\}$, the interpolating polynomial is*

$$p_n(f, \mathbf{x})=\sum_{|\alpha|=n} \bar{L}_\alpha f B_\alpha(\lambda),$$

where $\{\bar{L}_\alpha f\colon |\alpha|=n\}:=\{\bar{L}_{A_i^n\beta} f\colon \beta\in M_i, i=0, ..., s\}$ and each $\bar{L}_{A_i^n\beta} f$ is defined by

$$\bar{L}_{A_i^n\beta} f=\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}\frac{(n-|\gamma|)!}{n!} f_{i\gamma}, \qquad \beta\in M_i, \quad i=0, ..., s$$

In particular, if $\{f_{i\beta}=D_i^\beta f(\mathbf{v}^i)\colon \beta\in M_i, i=0, ..., s\}$, then

$$p_n(f, \mathbf{x})=\sum_{|\alpha|=n} L_\alpha f B_\alpha(\lambda).$$

Proof. Write $p_n(f, \mathbf{x})=\sum_{|\alpha|=n} c_\alpha B_\alpha(\lambda)$ as the interpolation polynomial. Then c_α can be obtained by using the dual functionals. That is,

$$c_{A_i^n\beta}=L_{A_i^n\beta}\, p_n=\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}\frac{(n-|\gamma|)!}{n!} D_i^\beta p_n(f, \mathbf{v}^i)$$

$$=\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}\frac{(n-|\gamma|)!}{n!} f_{i\beta}, \qquad \forall\beta\in M_i, \quad i=0, ..., s$$

since $p_n(f, \mathbf{x})$ interpolates the given data. Hence, we have established the theorem.

COROLLARY. *For each $\phi_\alpha(\mathbf{x})=\mathbf{x}^\alpha$, $|\alpha|\leqslant n$, its B-form is given by*

$$\phi_\alpha(\mathbf{x})=\sum_{|\gamma|=n} L_\gamma\phi_\alpha B_\gamma(\lambda).$$

Algorithm. To convert a given Taylor expansion of p_n (p_n in power form) into its B-form, we only need to compute its directional derivatives $D_i^\gamma p_n(\mathbf{v}^i)$, $\gamma\in M_i$, $i=0, ..., s$, and combine them in the following way

$$c_{A_i^n\beta}=\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}\frac{(n-|\gamma|)!}{n!} D_i^\gamma p_n(\mathbf{v}^i), \qquad \forall\beta\in M_i,$$

for $i=0, ..., s$. This gives all the B-coefficients c_α of $p_n(\mathbf{x})=\sum_{|\alpha|=n} c_\alpha B_\alpha(\lambda)$.

Alternatively, we may first find the B-coefficients $L_\gamma \phi_\alpha$, $|\gamma| = n$ of $\phi_\alpha(\mathbf{x}) = \mathbf{x}^\alpha$ as above for each $|\alpha| \leqslant n$. Then, for any polynomial $p_n(\mathbf{x}) = \sum_{|\alpha| \leqslant n} a_\alpha \mathbf{x}^\alpha$, the B-form for p_n is

$$\sum_{\substack{|\gamma| = n \\ \gamma \in \mathbb{Z}_+^{s+1}}} \left(\sum_{\substack{|\alpha| \leqslant n \\ \alpha \in \mathbb{Z}_+^s}} a_\alpha L_\gamma \phi_\alpha \right) B_\gamma(\lambda).$$

It is clear that these dual functionals are linear. In fact, they are bounded on $\mathbb{P}_n$:

THEOREM 3. *For integers n and $s \geqslant 1$, there exists a constant $C(n, s)$ dependent on n and s such that for any polynomial $p_n(\mathbf{x}) = \sum_{|\alpha| = n} c_\alpha B_\alpha$,*

$$|L_\gamma p_n| \leqslant C(n, s) \, \|p_n\|_\infty, \qquad |\gamma| = n,$$

where

$$C(n, s) = \frac{n!}{[n/(s+1)]!} (1 + 2^s)^{[sn/(s+1)]},$$

and $\|p_n\|_\infty = \max_{x \in T} \|p_n(\mathbf{x})\|$.

Proof. Let us consider $s = 1$ first. Then we can use the well-known Markov's inequality to get

$$|D_i^\beta p_n(\mathbf{v}^i)| \leqslant 2^\beta n^2 (n-1)^2 \cdots (n - \beta + 1)^2 \, \|p_n\|_\infty, \qquad \beta \leqslant n, \quad i = 0, 1.$$

Hence, let $M_0 = \{0, 1, ..., [(n-1)/2]\}$ and $M_1 = \{0, ..., [n/2]\}$. Then

$$\begin{aligned}
|c_{A_i^n \gamma}| = |L_{A_i^n \gamma} p_n| &= \left| \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} \frac{(n-\beta)!}{n!} D_i^\beta p_n(\mathbf{v}^i) \right| \\
&\leqslant \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} \frac{(n-\beta)!}{n!} |D_i^\beta p_n(\mathbf{v}^i)| \\
&\leqslant \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} \frac{(n-\beta)!}{n!} 2^\beta \left(\frac{n!}{(n-\beta)!} \right)^2 \|p_n\|_\infty \\
&= \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} \frac{n!}{(n-\beta)!} 2^\beta \, \|p_n\|_\infty \\
&\leqslant \frac{n!}{[n/2]!} \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} 2^\beta \, \|p_n\|_\infty \\
&= \frac{n!}{[n/2]!} (1+2)^\gamma \, \|p_n\|_\infty \\
&= \frac{n!}{[n/2]!} 3^\gamma \, \|p_n\|_\infty \leqslant \frac{n!}{[n/2]!} 3^{[n/2]} \, \|p_n\|_\infty, \qquad i = 0, 1.
\end{aligned}$$

Therefore, the result is true for the case $s = 1$.

Next, consider $s=2$. Let $M_0=\{(i,j): i+j\leqslant [2n/3]\}$. For each $\beta\in M_0$, we may apply Markov's inequality twice as follows:

$$
\begin{aligned}
|D_0^\beta p_n(\mathbf{v}^0)| &= |D_{01}^{\beta_1}(D_{02}^{\beta_2} p_n(\mathbf{v}^0))| \\
&\leqslant 4^{\beta_1} n^2 \cdots (n-\beta_1+1)^2 \max_{\mathbf{x}\in[\mathbf{v}^0,(\mathbf{v}^0+\mathbf{v}^1)/2]} |D_{02}^{\beta_2} p_n(\mathbf{x})| \\
&\leqslant 4^{\beta_1} n^2 \cdots (n-\beta_1+1)^2 |D_{02}^{\beta_2} p_n(\xi)|, \qquad \xi\in[\mathbf{v}^0, (\mathbf{v}^0+\mathbf{v}^1)/2] \\
&\leqslant 4^{\beta_1} n^2 \cdots (n-\beta_1+1)^2\, 4^{\beta_2}(n-\beta_1)^2 \cdots (n-\beta_1-\beta_2+1)^2 \\
&\quad \times \max_{\mathbf{x}\in[\xi,\xi+(\mathbf{v}^2-\mathbf{v}^1)]\cap T} |p_n(\mathbf{x})| \\
&= 4^{\beta_1+\beta_2} n^2 \cdots (n-|\beta_1+\beta_2|+1)^2 \|p_n\|_\infty,
\end{aligned}
$$

where $T=\langle \mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2\rangle$. Hence,

$$
\begin{aligned}
|c_{A_0^n\gamma}| = |L_{A_0^n\gamma} p_n| &\leqslant \sum_{\beta\leqslant\gamma} \binom{\gamma}{\beta} \frac{(n-|\beta|)!}{n!} |D_0^\beta p_n(\mathbf{v}^0)| \\
&= \sum_{\beta\leqslant\gamma} \binom{\gamma}{\beta} \frac{(n-|\beta|)!}{n!} 4^{|\beta|} \left(\frac{n!}{(n-|\beta|)!}\right)^2 \|p_n\|_\infty \\
&= \sum_{\beta\leqslant\gamma} \binom{\gamma}{\beta} \frac{n!}{(n-|\beta|)!} 4^{|\beta|} \|p_n\|_\infty \\
&= \frac{n!}{(n-|\gamma|)!} \|p_n\|_\infty \sum_{\beta\leqslant\gamma} \binom{\gamma}{\beta} 4^\beta \\
&\leqslant \frac{n!}{(n-[2n/3])!} (1+4)^{|\gamma|} \|p_n\|_\infty \leqslant \frac{n!}{[n/3]!} 5^{[2n/3]} \|p_n\|_\infty.
\end{aligned}
$$

Similarly, we will have estimates for $|c_{A_1^n\gamma}|$ and $|c_{A_2^n\gamma}|$. Hence, Theorem 3 is valid for $s=2$.

For the general case $s\geqslant 3$, we may apply Markov's inequality s times similar to the case $s=2$ and get $C(n,s)=(n!/[n/(s+1)]!)(1+2^s)^{[sn/(s+1)]}$ Thus, the proof is complete.

Remark. The existence of a constant C satisfying

$$
|c_\gamma| = |L_\gamma p_n| \leqslant C \|p_n\|_\infty, \qquad |\gamma|=n, \quad \forall p_n\in\mathbb{P}_n
$$

follows from the fact that $\mathbb{P}_n$ is finite-dimensional. Here, we give an explicit upper bound for C.

COROLLARY 1. *For any polynomial* $\sum_{|\alpha|=n} c_\alpha B_\alpha\in\mathbb{P}_n$,

$$
C(n,s)^{-1} \max_{|\alpha|=n} |c_\alpha| \leqslant \max_{\mathbf{x}\in T} \left| \sum_{|\alpha|=n} c_\alpha B_\alpha(\lambda) \right| \leqslant \max_{|\alpha|=n} |c_\alpha|.
$$

Next, we consider a bound on splines similar to Corollary 1. We need to introduce some more notation and definitions. Suppose that $\Omega \subseteq \mathbb{R}^s$ is a region partitioned by simplices. Thus, $\Omega = \bigcup_{i=1}^{T} t_i$, where each t_i is a simplex. Let $V = \{\mathbf{v}^1, ..., \mathbf{v}^N\}$ be the set of all vertices of Ω. Denote by

$$S_d^r(\Omega) = \{s \in C^r(\Omega): s\,|_{t_i} \in \mathbb{P}_d\}$$

the usual multivariate spline space. For each i, let $b_i(\mathbf{x}) \in S_1^0(\Omega)$ be the piecewise linear function satisfying $b_i(\mathbf{v}^j) = \delta_{ij}$, $j = 1, ..., N$. Clearly,

$$\mathbf{x} = \sum_{i=1}^{N} b_i(\mathbf{x})\,\mathbf{v}^i, \qquad \sum_{i=1}^{N} b_i = 1 \quad \text{and} \quad b_i \geqslant 0, \quad \forall i.$$

Set $A_d(\Omega) = \{\alpha \in \mathbb{Z}_+^N: \ |\alpha| = d, \ \langle \mathbf{v}^j: \alpha_j \neq 0 \rangle \subseteq t_i \text{ for some } t_i\}$. For each $\alpha \in A_d(\Omega)$ and $\lambda = (b_1(\mathbf{x}), ..., b_N(\mathbf{x}))$, we define

$$B_\alpha(\lambda, \Omega) = \frac{|\alpha|!}{\alpha!} \prod_{i=1}^{N} (b_i(\mathbf{x}))^{\alpha_i},$$

where $0^0 := 1$ the usual convention. Then each $s \in S_d^r(\Omega)$ may be expressed in the form

$$s(\mathbf{x}) = \sum_{\alpha \in A_d(\Omega)} c_\alpha B_\alpha(\lambda, \Omega)$$

which is called the spline in B-form with respect to Ω. This form of bivariate splines has been used successfully in [4].

We are now ready to state the following corollary.

COROLLARY 2. *For each* $s(\mathbf{x}) = \sum_{\alpha \in A_d(\Omega)} c_\alpha B_\alpha(\lambda, \Omega) \in S_d^0(\Omega)$,

$$C(d, s)^{-1} \max_{\alpha \in A_d(\Omega)} |c_\alpha| \leqslant \max_{\mathbf{x} \in \Omega} |s(\mathbf{x})| \leqslant \max_{\alpha \in A_d(\Omega)} |c_\alpha|.$$

Furthermore, if $T = \langle e^0, ..., e^s \rangle$ is the standard simplex in $\mathbb{R}^s$, then

$$\frac{\partial}{\partial x_j} p_n(\mathbf{x}) = n \sum_{|\alpha| = n-1} (c_{\alpha + A_0^n e^0} - c_{\alpha + A_0^n e^j})\, B_\alpha(\lambda),$$

for $\mathbf{x} = (x_1, ..., x_s) \in T$ and $j = 1, ..., s$. Thus,

$$\begin{aligned}
\left\| \frac{\partial}{\partial x_j} p_n \right\|_\infty &\leqslant n \max_{|\alpha| = n-1} |c_{\alpha + A_0^n e^0} - c_{\alpha + A_0^n e^j}| \\
&\leqslant 2n \max_{|\alpha| = n} |c_\alpha| \\
&\leqslant 2nC(n, s) \left\| \sum_{|\alpha| = n} c_\alpha B_\alpha \right\|_\infty \\
&= 2nC(n, s)\, \|p_n\|_\infty.
\end{aligned}$$

Therefore, we have proved the following theorem.

THEOREM 4. *Let T be the standard simplex. For each j, $1 \leqslant j \leqslant s$,*

$$\left\| \frac{\partial}{\partial x_j} p_n \right\|_\infty \leqslant 2nC(n, s) \|p_n\|_\infty.$$

Remark. Obviously, the constant $2nC(n, s)$ is not the best one and hence, the inequality in Theorem 4 is not the multivariate analog of the well-known Markov's inequality.

COROLLARY. *For any s-simplex T and for each $1 \leqslant j \leqslant s$,*

$$\left\| \frac{\partial}{\partial x_j} p_n \right\|_\infty \leqslant 2nC(n, s) \frac{\text{vol}_{s-1}\langle \mathbf{v}^0, ..., \mathbf{v}^{j-1}, \mathbf{v}^{j+1}, ..., \mathbf{v}^s \rangle}{\text{vol}_s \langle \mathbf{v}^0, ..., \mathbf{v}^s \rangle} \|p_n\|_\infty.$$

Proof. Since

$$D_{0k} p_n = \sum_{j=1}^{k} (\mathbf{v}^k - \mathbf{v}^0)_j \frac{\partial}{\partial x_j} p_n, \qquad k = 1, ..., s,$$

we may solve for $\partial/\partial x_j$, $j = 1, ..., s$, by using Cramer's rule and obtain

$$\frac{\partial}{\partial x_j} p_n = \frac{\det[(\mathbf{v}^1 - \mathbf{v}^0), ..., (\mathbf{v}^{j-1} - \mathbf{v}^0), D, (\mathbf{v}^{j+1} - \mathbf{v}^0), ..., (\mathbf{v}^s - \mathbf{v}^0)]}{\det[(\mathbf{v}^1 - \mathbf{v}^0), ..., (\mathbf{v}^s - \mathbf{v}^0)]},$$

where D is a column vector $(D_{01} p_n, ..., D_{0s} p_n)^t$. After a simple expansion of the determinant, we obtain

$$\frac{\partial}{\partial x_j} p_n = \frac{\begin{pmatrix} \sum_{k=1}^s D_{0k}(-1)^k P_k \langle (\mathbf{v}^1 - \mathbf{v}^0), ..., (\mathbf{v}^{j-1} - \mathbf{v}^0), \\ \qquad (\mathbf{v}^{j+1} - \mathbf{v}^0), ..., (\mathbf{v}^s - \mathbf{v}^0) \rangle \end{pmatrix}}{s! \, \text{vol}_s \langle \mathbf{v}^0, ..., \mathbf{v}^s \rangle},$$

where $P_k \langle (\mathbf{v}^1 - \mathbf{v}^0), ..., (\mathbf{v}^{j-1} - \mathbf{v}^0), (\mathbf{v}^{j+1} - \mathbf{v}^0), ..., (\mathbf{v}^s - \mathbf{v}^0) \rangle$ denotes the projection of $\langle (\mathbf{v}^1 - \mathbf{v}^0), ..., (\mathbf{v}^{j-1} - \mathbf{v}^0), (\mathbf{v}^{j+1} - \mathbf{v}^0), ..., (\mathbf{v}^s - \mathbf{v}^0) \rangle$ onto the coordinate plane which is perpendicular to the kth axis. Hence,

$$\left| \frac{\partial p_n}{\partial x_j} \right| \leqslant \frac{\begin{pmatrix} (s-1)! \, \text{vol}_{s-1} \langle (\mathbf{v}^1 - \mathbf{v}^0), ..., (\mathbf{v}^{j-1} - \mathbf{v}^0), \\ \qquad (\mathbf{v}^{j+1} - \mathbf{v}^0), ..., (\mathbf{v}^s - \mathbf{v}^0) \rangle \end{pmatrix}}{s! \, \text{vol}_s \langle \mathbf{v}^0, ..., \mathbf{v}^s \rangle} \sum_{k=1}^{s} |D_{0k} p_n|.$$

This completes the proof.

It is already known that

$$\max_{|\alpha| = n} |p_n(\mathbf{x}_\alpha) - c_\alpha| \leqslant K(\text{diam } T)^2 \max_{|\beta| = 2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} p_n \right\|_\infty,$$

where $\mathbf{x}_\alpha = (1/n) \sum_{i=0}^{s} \alpha_i \mathbf{v}^i$, $\forall\ |\alpha| = n$, and diam T denotes the diameter of T (cf., e.g., [9]). This estimate is crucial to give an *a priori* estimate before applying the B-net subdivision algorithm. We are now able to specify the K in the following theorem.

THEOREM 5. *For any n, s and polynomial $p_n(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha B_\alpha(\lambda) \in \mathbb{P}_n$,*

$$\max_{|\alpha|=n} |p_n(\mathbf{x}_\alpha) - c_\alpha| \leqslant \frac{C(n, s)}{2n} s(E(T))^2 \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} p_n \right\|_\infty,$$

where $C(n, s)$ is defined in Theorem 3 and $E(T)$ denotes the longest length of edges of T, i.e., $E(T) := \max_{i,j} \|\mathbf{v}^i - \mathbf{v}^j\|_2$.

Proof. It is known that for any $\phi \in \mathbb{P}_1$,

$$\phi(\mathbf{x}) = \sum_{|\alpha|=n} \phi(\mathbf{x}_\alpha) B_\alpha(\lambda).$$

(Refer to [3, 9].) For any function f,

$$f(\mathbf{y}) - f(\mathbf{x}) = D_{\mathbf{y}-\mathbf{x}} f(\mathbf{x}) + \tfrac{1}{2} D^2_{\mathbf{y}-\mathbf{x}} f(\mathbf{x} + \xi(\mathbf{y} - \mathbf{x})),$$

for some $\xi \in [0, 1]$. Thus, we get

$$\sum_{|\alpha|=n} f(\mathbf{x}_\alpha) B_\alpha(\lambda) - f(\mathbf{x}) = \frac{1}{2} \sum_{|\alpha|=n} D^2_{\mathbf{x}_\alpha - \mathbf{x}} f(\mathbf{x} + \xi(\mathbf{x}_\alpha - \mathbf{x})) B_\alpha(\lambda).$$

Clearly,

$$|D^2_{\mathbf{x}_\alpha - \mathbf{x}} f(\mathbf{x} + \xi(\mathbf{x}_\alpha - \mathbf{x}))| \leqslant s \left(\sum_{i=1}^{s} |(\mathbf{x}_\alpha - \mathbf{x})_i|^2 \right) \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty$$

Hence,

$$\begin{aligned}
&\left| f(\mathbf{x}) - \sum_{|\alpha|=n} f(\mathbf{x}_\alpha) B_\alpha(\lambda) \right| \\
&\quad \leqslant \frac{s}{2} \left| \sum_{|\alpha|=n} \sum_{i=1}^{s} |(\mathbf{x}_\alpha - \mathbf{x})_i|^2 B_\alpha(\lambda) \right| \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty \\
&\quad = \frac{s}{2} \sum_{|\alpha|=n} \frac{1}{n^2} \sum_{j=1}^{s} \sum_{i,k=0}^{s} \alpha_i (\mathbf{v}^i - \mathbf{x})_j \alpha_k (\mathbf{v}^k - \mathbf{x})_j B_\alpha(\lambda) \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty \\
&\quad = \frac{s}{2} \frac{1}{n^2} \sum_{j=1}^{s} \sum_{i,k=0}^{s} (\mathbf{v}^i - \mathbf{x})_j (\mathbf{v}^k - \mathbf{x})_j \sum_{|\alpha|=n} \alpha_i \alpha_k B_\alpha(\lambda) \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty
\end{aligned}$$

Since

$$\sum_{|\alpha|=n} \alpha_i \alpha_k B_\alpha(\lambda) = \begin{cases} n(n-1)\,\lambda_i \lambda_k, & i \neq k \\ n(n-1)\,\lambda_i^2 + n\lambda_i, & i = k, \end{cases}$$

$$\begin{aligned} &\left| f(\mathbf{x}) - \sum_{|\alpha|=n} f(\mathbf{x}_\alpha)\, B_\alpha(\lambda) \right| \\ &\leqslant \frac{s}{2}\left[\frac{n-1}{n} \sum_{j=1}^{s} \left(\sum_{i=0}^{s} \lambda_i (\mathbf{v}^i - \mathbf{x})_j \right) \left(\sum_{k=0}^{s} \lambda_k (\mathbf{v}^k - \mathbf{x})_j \right) \right. \\ &\quad \left. + \frac{1}{n} \sum_{j=1}^{s} \sum_{i=0}^{s} \lambda_i (\mathbf{v}^i - \mathbf{x})_j^2 \right] \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty \\ &= \frac{s}{2n} \left| \sum_{i=0}^{s} \lambda_i \mathbf{v}^i \cdot \mathbf{v}^i - \mathbf{x} \cdot \mathbf{x} \right| \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty \\ &\leqslant \frac{s}{2n} \max_{\mathbf{x} \in T} \left| \sum_{i=0}^{s} \lambda_i (\mathbf{v}^i - \mathbf{x}) \cdot (\mathbf{v}^i - \mathbf{x}) \right| \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty \\ &\leqslant \frac{s}{2n} E(T)^2 \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} f \right\|_\infty . \end{aligned}$$

Therefore, by Theorem 3

$$\begin{aligned} |p_n(\mathbf{x}_\alpha) - c_\alpha| &\leqslant C(n, s) \left\| \sum_{|\alpha|=n} (p_n(\mathbf{x}_\alpha) - c_\alpha)\, B_\alpha \right\|_\infty \\ &= C(n, s) \left\| \sum_{|\alpha|=n} p_n(\mathbf{x}_\alpha)\, B_\alpha - p_n \right\|_\infty \\ &\leqslant C(n, s) \frac{s}{2n} E(T)^2 \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} p_n \right\|_\infty \end{aligned}$$

The proof is complete.

Next, we would like to know how fast the B-nets arising from successive degree raising approximate the polynomial surface. Recall that for any polynomial p_n in B-form, it can be rewritten as a polynomial of higher degree, i.e.,

$$p_n(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha B_\alpha(\lambda) = \sum_{|\alpha|=n+l} R^l c_\alpha B_\alpha(\lambda),$$

where

$$R^l c_\alpha = \sum_{\gamma \leqslant \alpha} c_\gamma \binom{\alpha}{\gamma} \Big/ \binom{n+l}{n}.$$

(Cf. [3, 9, 12].)

It was proved in [10] and in many other papers that

$$\max_{|\alpha|=n+l} |p_n(\mathbf{x}_\alpha) - R^l c_\alpha| \to 0, \qquad l \to \infty.$$

Here, we prove the following theorem.

THEOREM 6. *Let T be the standard simplex in $\mathbb{R}^s$. For any $p_n \in \mathbb{P}_n$, for l sufficiently large,*

$$\max_{|\alpha|=n+l} |p_n(\mathbf{x}_\alpha) - R^l c_\alpha| \leqslant \frac{1}{(n+l)} \frac{2sn^2 C(n,s)^2}{1 - 2sn^2 C(n,s)^2/(n+l)} \|p_n\|_\infty.$$

Proof. Define the operator B_k on $\mathbb{P}_n$ by

$$B_k p_n(\mathbf{x}) = \sum_{|\alpha|=k} p_n(\mathbf{x}_\alpha) B_\alpha(\lambda), \qquad \forall p_n \in \mathbb{P}_n.$$

By Lemma 1, we know that $B_k p_n \subseteq \mathbb{P}_n$, when $k \geqslant n$. As shown in the proof of Theorem 5,

$$\begin{aligned}\|B_k p_n - p_n\|_\infty &\leqslant \frac{s}{2k} \max_{|\beta|=2} \left\| \frac{\partial^2}{\partial \mathbf{x}^\beta} p_n \right\|_\infty \\ &\leqslant \frac{2sn^2 C(n,s)^2}{k} \|p_n\|_\infty\end{aligned}$$

by using Theorem 4.

When k is large enough,

$$\|B_k p_n - p_n\|_\infty \leqslant \frac{2sn^2 C(n,s)^2}{k} \|p_n\|_\infty < \|p_n\|_\infty, \qquad \forall p_n \in \mathbb{P}_n.$$

Thus, B_k is invertible on $\mathbb{P}_n$ and hence,

$$\begin{aligned}&\max_{|\alpha|=n+l} |R^l c_\alpha - p_n(\mathbf{x}_\alpha)| \\ &\quad = \max_{|\alpha|=n+l} |(B_{n+l}^{-1} p_n)(\mathbf{x}_\alpha) - p_n(\mathbf{x}_\alpha)| \\ &\quad \leqslant \|B_{n+l}^{-1} p_n - p_n\|_\infty \leqslant \|B_{n+l}^{-1} - I\|_\infty \|p_n\|_\infty \\ &\quad = \|[I + (B_{n+l} - I)]^{-1} - I\|_\infty \|p_n\|_\infty \\ &\quad \leqslant \frac{\|B_{n+l} - I\|_\infty}{1 - \|B_{n+l} - I\|_\infty} \|p_n\|_\infty \leqslant \frac{1}{(n+l)} \frac{2sn^2 C(n,s)^2}{1 - 2sn^2 C(n,s)^2/(n+l)} \|p_n\|_\infty.\end{aligned}$$

This completes the proof.

COROLLARY. *If T is an arbitrary s-simplex in $\mathbb{R}^s$, then for any $p_n \in \mathbb{P}_n$,*

$$\max_{|\alpha| = n+l} |p_n(\mathbf{x}_\alpha) - R^l c_\alpha| \leqslant \frac{1}{n+l} \frac{2sn^2 C(n, s)^2\, h(T)^2}{1 - 2sn^2 C(n, s)^2/(n+l)\, h(T)^2} \| p_n\|_\infty,$$

where

$$h(T) = E(T) \max_{0 \leqslant i \leqslant s} \left\{ \frac{\mathrm{Vol}_{s-l}\langle \mathbf{v}^0, ..., \mathbf{v}^{i-1}, \mathbf{v}^{i+1}, ..., \mathbf{v}^s \rangle}{\mathrm{Vol}_s \langle \mathbf{v}^0, ..., \mathbf{v}^s \rangle} \right\}.$$

Alternatively, we may use these dual linear functionals to give the following estimate in terms of the B-coefficients of $p_n(\mathbf{x})$. This avoids using the (unknown) multivariate version of well-known Markov's inequality and the multivariate version of Bernstein polynomial convergence theorem.

THEOREM 7. *For any $p_n \in \mathbb{P}_n$ with B-coefficients $\{c_\beta\}_{|\beta| = n}$, for $|\alpha| = n + l$,*

$$|p_n(\mathbf{x}_\alpha) - R^l c_\alpha| \leqslant \frac{n(n-1)}{n+l} (1 + 2s)^n \, \|\{c_\beta\}_{|\beta| = n}\|_\infty + O\left(\frac{1}{(n+l)^2}\right), \qquad \textit{as} \quad l \to +\infty.$$

Proof. Let c_0 be a mapping defined by

$$c_0 \alpha = (\alpha_1, ..., \alpha_s), \qquad \text{for} \quad \alpha = (\alpha_0, \alpha_1, ..., \alpha_s) \in \mathbb{Z}_+^{s+1}.$$

We first expand $p_n(\mathbf{x}_\alpha)$ at $\mathbf{v}^0$ to obtain

$$\begin{aligned}
p_n(\mathbf{x}_\alpha) &= \sum_{j=0}^{n} \frac{1}{j!} (D_{\mathbf{x}_\alpha - \mathbf{v}^0})^j p_n(\mathbf{v}^0) \\
&= \sum_{j=0}^{n} \frac{1}{j!} \left[\sum_{i=1}^{s} \frac{\alpha_i}{n+l} D_{i0} \right]^j p_n(\mathbf{v}^0) \\
&= \sum_{j=0}^{n} \sum_{|\gamma| = j} \frac{1}{\gamma!} \frac{(c_0\alpha)^\gamma}{(n+l)^j} D_0^\gamma p_n(\mathbf{v}^0) \\
&= p_n(\mathbf{v}^0) + \sum_{i=1}^{s} \frac{\alpha_i}{n+l} D_{i0} p_n(\mathbf{v}^0) \\
&\quad + \sum_{j=2}^{n} \left(\sum_{\substack{|\gamma| = j \\ \gamma \leqslant c_0\alpha}} + \sum_{\substack{|\gamma| = j \\ \exists i, \gamma_i > \alpha_i}} \right) \frac{1}{\gamma!} \frac{(c_0\alpha)^\gamma}{(n+l)^j} D_0^\gamma p_n(\mathbf{v}^0) \\
&= p_n(\mathbf{v}^0) + \sum_{i=1}^{s} \frac{\alpha_i}{n+l} D_{i0} p_n(\mathbf{v}^0) \\
&\quad + \sum_{j=2}^{n} \sum_{\substack{|\gamma| = j \\ \gamma \leqslant c_0\alpha}} \frac{1}{\gamma!} \frac{(c_0\alpha)^\gamma}{(n+l)^j} \frac{n!}{(n-j)!} \Delta_0^\gamma c_{(n-j)e^0} + O\left(\frac{1}{(n+l)^2}\right).
\end{aligned}$$

Then we use the dual functional formula to get

$$R^l c_\alpha = \sum_{\gamma \leqslant c_0\alpha} \binom{c_0\alpha}{\gamma} \frac{(n+l-|\gamma|)!}{(n+l)!} D_0^\gamma p_n(\mathbf{v}^0)$$

$$= p_n(\mathbf{v}^0) + \sum_{i=1}^{s} \frac{\alpha_i}{n+l} D_{i0}\, p_n(\mathbf{v}^0)$$

$$+ \sum_{j=2}^{n} \sum_{\substack{|\gamma|=j \\ \gamma \leqslant c_0\alpha}} \binom{c_0\alpha}{\gamma} \frac{(n+l-j)!}{(n+l)!} D_0^\gamma p_n(\mathbf{v}^0)$$

$$= p_n(\mathbf{v}^0) + \sum_{i=1}^{s} \frac{\alpha_i}{n+l} D_{i0}\, p_n(\mathbf{v}^0)$$

$$+ \sum_{j=2}^{n} \sum_{\substack{|\gamma|=j \\ \gamma \leqslant c_0\alpha}} \frac{1}{\gamma!} \frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!} \frac{(n+l-j)!}{(n+l)!} \frac{n!}{(n-j)!} \Delta_0^\gamma c_{(n-j)e^0}.$$

Thus,

$$p_n(\mathbf{x}_\alpha) - R^l c_\alpha = \sum_{j=2}^{n} \sum_{\substack{|\gamma|=j \\ \gamma \leqslant c_0\alpha}} \frac{1}{\gamma!} \left[\frac{(c_0\alpha)^\gamma}{(n+l)^j} - \frac{(n+l-j)!}{(n+l)!} \frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!} \right]$$

$$\times \frac{n!}{(n-j)!} \Delta_0^\gamma c_{(n-j)e^0} + O\left(\frac{1}{(n+l)^2} \right).$$

Since

$$\frac{(n+l-j)!}{(n+l)!} = \frac{1}{(n+l)^j} \frac{1}{\prod_{k=1}^{j-1} (1-k/(n+l))}$$

$$= \frac{1}{(n+l)^j} \prod_{k=1}^{j-1} \left(1 + \frac{k}{n+l} + O\left(\frac{1}{(n+l)^2} \right) \right)$$

$$= \frac{1}{(n+l)^j} \left(1 + \sum_{k=1}^{j-1} \frac{k}{n+l} + O\left(\frac{1}{(n+l)^2} \right) \right)$$

$$= \frac{1}{(n+l)^j} + \frac{(j-1)j}{2} \frac{1}{(n+l)^{j+1}} + O\left(\frac{1}{(n+l)^{j+2}} \right),$$

$$\frac{(n+l-j)!}{(n+l)!} \frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!} = \frac{1}{(n+l)^j} \frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!}$$

$$+ \frac{j(j-1)}{2} \frac{1}{(n+l)^{j+1}} \frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!} + O\left(\frac{1}{(n+l)^2} \right).$$

Also, since

$$\frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!}=(c_0\alpha)^\gamma\prod_{i=1}^{s}\left(1-\frac{1}{\alpha_i}\right)\cdots\left(1-\frac{\gamma_i-1}{\alpha_i}\right)$$
$$=(c_0\alpha)^\gamma\left(1-\sum_{i=1}^{s}\sum_{k=1}^{\gamma_i-1}\frac{k}{\alpha_i}+\cdots\right)$$
$$=(c_0\alpha)^\gamma-\sum_{i=1}^{s}\frac{\gamma_i(\gamma_i-1)}{2}(c_0\alpha)^{\gamma-e^i}+O((n+l)^{j-2}),$$

$$\frac{(c_0\alpha)^\gamma}{(n+l)^j}-\frac{(n+l-j)!}{(n+l)!}\frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!}$$
$$=\frac{(c_0\alpha)^\gamma}{(n+l)^j}-\left[\frac{(c_0\alpha)^\gamma}{(n+l)^j}+\frac{j(j-1)}{2}\frac{(c_0\alpha)^\gamma}{(n+l)^{j+1}}\right.$$
$$\left.-\sum_{i=1}^{s}\frac{\gamma_i(\gamma_i-1)}{2}\frac{(c_0\alpha)^{\gamma-e^i}}{(n+l)^j}+O\left(\frac{1}{(n+l)^2}\right)\right].$$

Thus,

$$\left|\frac{(c_0\alpha)^\gamma}{(n+l)^j}-\frac{(n+l-j)!}{(n+l)!}\frac{(c_0\alpha)!}{(c_0\alpha-\gamma)!}\right|$$
$$\leqslant\sum_{i=1}^{s}\frac{\gamma_i(\gamma_i-1)}{2}\frac{1}{n+l}+\frac{j(j-1)}{2}\frac{1}{n+l}+O\left(\frac{1}{(n+l)^2}\right)$$
$$\leqslant\frac{1}{n+l}\left(\frac{j^2-j}{2}+\frac{j(j-1)}{2}\right)+O\left(\frac{1}{(n+l)^2}\right)$$
$$=\frac{j(j-1)}{n+l}+O\left(\frac{1}{(n+l)^2}\right).$$

Hence,

$$|p_n(\mathbf{x}_\alpha)-R^l c_\alpha|$$
$$\leqslant\sum_{j=2}^{n}\sum_{|\gamma|=j}\frac{1}{\gamma!}\frac{j(j-1)}{n+l}\frac{n!}{(n-j)!}|\Delta_0^\gamma c_{(n-j)e^0}|+O\left(\frac{1}{(n+l)^2}\right)$$
$$=\frac{1}{n+l}\sum_{j=2}^{n}\frac{1}{(j-2)!}\left(\sum_{|\gamma|=j}\frac{j!}{\gamma!}\right)\frac{n!}{(n-j)!}2^j\,\|\{c_\beta\}_{|\beta|=n}\|_\infty+O\left(\frac{1}{(n+l)^2}\right)$$
$$=\frac{1}{n+l}\sum_{j=2}^{n}\frac{1}{(j-2)!}(2s)^j\frac{n!}{(n-j)!}\|\{c_\beta\}_{|\beta|=n}\|_\infty+O\left(\frac{1}{(n+l)^2}\right)$$
$$=\frac{n(n-1)}{n+l}(1+2s)^{n-2}(2s)^2\,\|\{c_\beta\}_{|\beta|=n}\|_\infty+O\left(\frac{1}{(n+l)^2}\right)$$
$$\leqslant\frac{n(n-1)}{n+l}(1+2s)^n\,\|\{c_\beta\}_{|\beta|=n}\|_\infty+O\left(\frac{1}{(n+l)^2}\right).$$

This completes the proof.

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